

SR: The laws of physics are invariant under transformations connecting inertial frames, and spacetime is isotropic in space and homogenous in space-time. $\Rightarrow P^4 = ISO(1,3)^\uparrow \times T^4$

GR: The laws of physics are invariant under diffeomorphisms of spacetime } tells us how things behave on curved space, i.e. respond to gravity.
and the connection (gauge-field) facilitating this invariance should be rendered dynamical by the introduction of an invariant field strength term. } tells us how curvature arises dynamically from sources

It will take time to digest all of this, especially the second half, but the first half will seem similar to what we did in SR, i.e. identify transformations and then demand laws be expressed in terms of true tensors under those transformations.

To get an idea of why the transformations in GR will be more complicated, consider SR where we can coordinatize M^4 globally w/ (t, x, y, z) and metric $\eta_{\mu\nu}$. This means that the Lorentz transformations (actually Poincaré) act the same everywhere, and it is in these coordinates that physical laws take their simplest form (preferred coordinates).

However we learned from the EEP that when gravity is present, we will at best be able to describe things in terms of flat space physics over a small region. We will not be able to extend this to a global symmetry of the entire space. Moreover, since each region can be made flat exclusively of the others, there is no preferred coordinate system, so we are going to need a much more general set of transformations that exhibit no preferred coordinates. \Rightarrow "General coordinate transformations" or "General Covariance"

But coordinates are really not physical, so much of this we should be able to consider in a coordinate independent form.

Einstein Equivalence Principle: Experiments performed over a short time in a small freely-falling lab give results indistinguishable from those obtained in an inertial frame in empty space.

This along with the absence of any gravitationally "neutral" objects lead us to define "inertial" frames as freely-falling frames (FFF) since:

$$\boxed{\text{Lab}} \downarrow \vec{g} = \boxed{\text{Lab}} \uparrow \vec{a} \Rightarrow \boxed{\text{Lab}} \downarrow \vec{g} - \boxed{\text{Lab}} \uparrow \vec{a} = \boxed{\text{Lab}} \downarrow \vec{a} \text{ freely-falling}$$

uniform

Small labs and short times are required to ensure approximate uniformity of \vec{g} , even if it changes w/ position.

2 important implications:

1. In an FFF we can use SR since this correctly describes physics in deep space (no gravity).
2. When looking for what type of spacetimes are allowed in GR, they must be locally flat, i.e. a manifold.

Nondegenerate:	$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$ Euclidean Space (in x_1, x_2, \dots)	} Special Cases
	$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$ Minkowski Space (in t, x_1, x_2, \dots)	
Riemannian	$g_{\mu\nu} = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$ w/ $a, b, c, \dots > 0 \Rightarrow$ Euclidean Signature	
pseudo-Riemannian	$g_{\mu\nu} = \begin{pmatrix} -a & & \\ & b & \\ & & c \end{pmatrix}$ w/ $a, b, c, \dots > 0 \Rightarrow$ Lorentzian Signature	

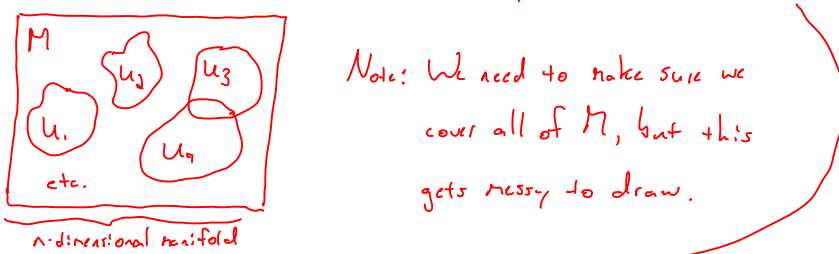
So what is a manifold?

A C^p n -dimensional manifold is a set M with a maximal atlas.

That's nice, but what do C^p , maximal and atlas mean?

The set M we have in mind is the collection of points in space (time).

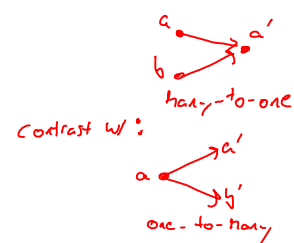
Let's begin by breaking M up into patches called charts:



A chart is a subset of M with a one-to-one map $\phi: U_\alpha \rightarrow \mathbb{R}^n$ such that the image of ϕ is open in \mathbb{R}^n .

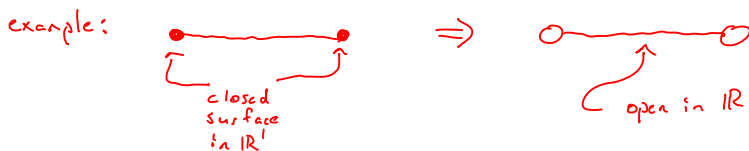
This notation means ϕ is a map from U_α to \mathbb{R}^n (similar to a function)

Okay, but what does one-to-one and open mean?

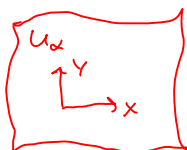


Aside: one-to-one + onto = bijective or invertible
injective surjective

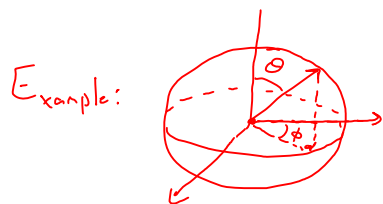
open: Open in \mathbb{R}^n means the interior of an $(n-1)$ -dimensional closed surface



Okay, these maps which define charts may sound abstract, but they really just boil down to our ability to choose "good" coordinates for the patch U_α .



Takes a point in U_α and sends it to $(x, y) \in \mathbb{R}^2$



Example: (θ, ϕ) coordinates for a curved S^2 take values in \mathbb{R}^2 .

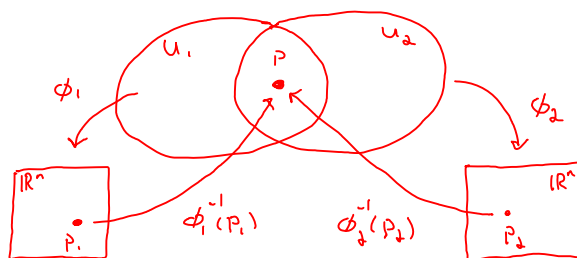
Now that we know what a chart is, we can define an atlas:

An atlas on M is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ such that;

- a) The union of all charts U_α is equal to M .
- b) The charts are sewn together w/ C^p transition functions.

Okay, wait, what are C^p transition functions?

Suppose two charts overlap:



Then: $p_1 = \phi_1(p) = \phi_1 \circ \phi_2^{-1}(p_2) : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$p_2 = (x_1, y_1, z_1, \dots)$

$p_1 = (x_2, y_2, z_2, \dots)$

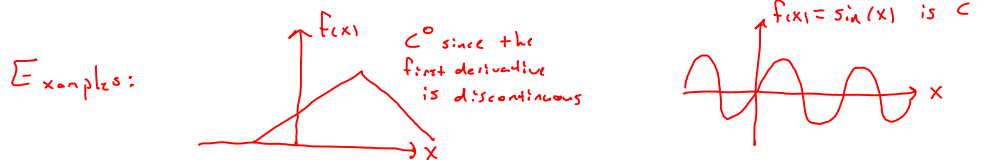
$p_1, p_2 \in \mathbb{R}^n$

The transition map or transition function $\phi_{12} = \phi_1 \circ \phi_2^{-1}$ gives $\begin{cases} x_1 = f_x(x_2, y_2, \dots) \\ y_1 = f_y(x_2, y_2, \dots) \\ \vdots \end{cases}$

But these are just coordinate changes!

Now that we are dealing with functions from \mathbb{R}^n to \mathbb{R}^n we can talk about calculus and in particular derivatives.

If the p th-derivative of all $\phi_{\alpha\beta}$ exists and is continuous, then $\phi_{\alpha\beta}$ is C^p .
If $p = \infty$, i.e. C^∞ then we call $\phi_{\alpha\beta}$ smooth.



For an example of a C^1 map, we can just use $\int f(x) dx$ where f is C^0 , etc.

If the C^∞ map $\phi_{\alpha\beta}$ is also invertible then we call it a diffeomorphism.

The coordinate transformations we will encounter will almost always be diffeomorphisms.

Last but not least we have to define maximal.

A maximal atlas is an equivalence class. This basically means that if I take one manifold, I may be able to cover w/ different choices of atlases. But each choice is an equivalently good representation of the manifold so they belong to the maximal atlas.

An important consequence of maximal is that the C^p condition must hold on all atlases, even when two charts completely overlap, i.e. C^p is required for all regions in M !

Going back: A C^p n -dimensional manifold is a set M w/ a maximal atlas.

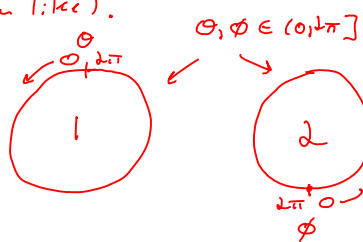
Let's put all this fancy math to use with an example.

Prove that S^1 (a circle) is a manifold.

To prove S^1 is a manifold we need only construct one representative element from the maximal atlas.

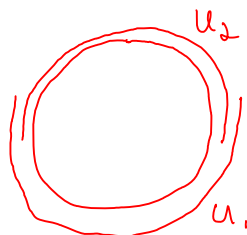
S^1 is 1D. You might think we could cover it w/ 1 chart, but it turns out we need at least 2 (you could use more if you like).

Let's label the points in our set M in 2 ways:



It may seem like we just coordinatized the S^1 (mapped it to \mathbb{R}^1), but remember we need an open subset of \mathbb{R}^1 and $(0, 2\pi]$ is not (it contains 2π).

So we use 2 charts as follows:



w/ the coordinate systems above

$$\begin{array}{c}
 (\frac{\pi}{2} - \epsilon, \frac{3\pi}{2} + \epsilon) \\
 \downarrow \\
 U_1 \text{ w/ } \phi_1: \theta \rightarrow x_1 \text{ or } \theta = x_1 \\
 U_2 \text{ w/ } \phi_2: \phi \rightarrow x_2 \text{ or } \phi = x_2 \\
 \uparrow \\
 (\frac{\pi}{2} - \epsilon, \frac{3\pi}{2} + \epsilon)
 \end{array}$$

The maps are so trivial that it is easy to see that they are one-to-one and image is open in \mathbb{R}^1 .

Now we need to check that they form an atlas.

1. Do they combine to cover all of S^1 ? Yes
2. Are the transition functions C^p for some p ? Consider the overlap

The transition function $\phi_1 \circ \phi_2^{-1}(x_2) = x_1$

is then $\phi_{21}: x_1(x_2) = x_2 + \pi$ which is C^∞ !

We can do something similar for the other overlap.

